QUAID-I-AZAM UNIVERSITY

DEPARTMENT OF MATHEMATICS

M.Phil. Admission Test Fall 2010

Time: 90 minutes

Dated: 18-08-2010

Note: Section I is compulsory, Section II is for Applied Mathematics and Section III for Pure Mathematics candidates.

Use separate page for each question.

Section I

Q. 1. If $0 < a < b$, determine $\lim\limits_{n\to\infty} \left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)$.

Q. 2. Compute the integral or prove its divergence $\int\limits_0^1 \frac{dx}{\sqrt{1-x^2}}$.

Q. 3. a) In the group $(\mathbb{C}-\{0\},\cdot)$ mention an element with order n.

b) In the group $(\mathbb{C}-\{0\},\cdot)$ mention an element with order ∞.

c) When a group is of finite order?

d) Give an example of a multiplicative group with order 8 containing a prime field with 3 elements.

e) Establish an isomorphism between the groups $(\mathbb{R},+)$ and $(\mathbb{R}^+,\cdot)$.

Q. 4. Let $X \neq \phi$ and $x_0 \in X$. Show that $\Im = \{X\} \cup \{A \subseteq X : x_0 \notin A\}$ forms a topology on X.

Q. 5. Solve $x^2y^3 + x(1+y^2)y' = 0$.

Q. 6. Let T be a linear operator on $\mathbb{R}^3$ defined by $T(x,y,z) = (2y+z, x-4y, 3x)$. Find the matrix of T in the basis $\{(1,1,1),(1,1,0),(1,0,0)\}$.

Q. 7. Define a conservative force. Prove that the work done by such a force around a closed loop is zero. Is the converse true?

Q. 8. Let C denote the circle $|z| = 1$, taken in the positive sense. Evaluate the integral $\int\limits_C \exp(z+\frac{1}{z})dz$.

Q. 9. Find the first fundamental form of the surface:

$x = \{3(\cos\phi\sin\theta), 3(\sin\phi\sin\theta), 3(\cos\theta)\} \qquad 0 \le \theta \le \pi, 0 \le \phi \le 2\pi.$

Q. 10. Find the contracted tensor components of F^a_b, where

$F^{ac}_{db} = A^a_d B^c_b$ and $A^a_d = \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$, $B^c_b = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 0 & 9 \end{bmatrix}$.

Q. 11. Show that $\left(\frac{\Delta^2}{E}\right)(e^x)\cdot\frac{E(e^x)}{\Delta^2(e^x)} = e^x$.

Q. 12. Solve $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t};\ o < x < \infty, t > 0$

$u(0,t) = A,\ t \ge 0$

$u(x,0) = 0,\ 0 < x < \infty$

u and $\frac{\partial u}{\partial x}$ both tend to zero when $x \to \infty$.

Q. 13. Let $T : N \to M$ be a surjective bounded and linear operator from a normed space N to a normed space M. If there exists a positive real number b such that $\|Tx\| \ge b\|x\|$ for all $x \in N$, then T^{-1} exists and is bounded.

Section II

Q. 14. Find the solution of $\operatorname{Si} nx = \int_0^x e^{x-t}u(t)dt$.

Q. 15. Is the motion $u = \frac{kx}{x^2+y^2}, v = \frac{ky}{x^2+y^2}, w = 0$ kinematically possible for an incompressible fluid (k is constant).

Q. 16. Write the Lorentz transformations as a rotation about a fixed axis, hence prove that the Lorentz transformations form a rotation group (Lorentz -Poincare group).

Q. 17. Let $\widehat{A}$ be an operator such that for a state vector $|\Psi\rangle$:

$|\Psi\rangle = 2i\,|+\rangle - 3\,|0\rangle + |-\rangle$,

$\widehat{A}\,|\Psi\rangle = 3\,|+\rangle + |0\rangle - i\,|-\rangle$.

Find the matrix and outer product representation for $\widehat{A}$ in basis $\{|+\rangle, |0\rangle, |-\rangle\}$.

Q. 18. Derive equation of motion for isotropic elastic medium.

Q. 19. Derive wave equations for conducting medium in terms of electric and magnetic fields.

Q. 20. Use Picard's method to approximate $y(0.1)$ given that $\frac{dy}{dx} = 1 + xy$, $y(0) = 1$. By performing four iterations.

Section III

Q. 21. Let H be a Hilbert space and $A \subseteq H$. Then $\left(\overline{A}\right)^{\perp} = A^{\perp}$.

Q. 22. In how many ways can 7 boys and 2 girls be lined up in a row such that the girls must be separated by exactly 3 boys?

Q. 23. a) Prove that $\frac{\mathbb{Z}}{n\mathbb{Z}} \simeq \mathbb{Z}_n$.

b) Verify that F^n is an algebra over the field F.

c) For field F verify that if $k < n$ be positive integers, then

$0 \to F^k \xrightarrow{\sigma} F^n \xrightarrow{\xi} F^{n-k} \to 0$ is short exact sequence of F-algebras.

(Hint show that $\operatorname{Im}\sigma = \ker\xi$) (3+3+4)

Q. 24. a) Define a linear code.

b) In short exact sequence of $\mathbb{Z}_2$-spaces

$0 \to \mathbb{Z}_2^k \to \mathbb{Z}_2^n \to \mathbb{Z}_2^{n-k} \to 0$ explain the code is linear subspace of $\mathbb{Z}_2^n$ having dimension k but each code vector has n digits.

c) What is a cyclic code?

d) Code described in (b) has a specific name, explain. (2+3+2+3)

Q. 25. Show that retract of a projective module is projective.

Q. 26. Show that outer measure of an interval is its length.

Q. 27. Prove that every action gives rise to a permutation representation and vice versa.

Good Luck

Sp-2010
P(1)

QUAID-I-AZAM UNIVERSITY

DEPARTMENT OF MATHEMATICS

M.Phil. Admission test Spring 2010

Time: 90 minutes

Dated: 02-02-2010

Note: Section I is compulsory, Section II is for Applied Mathematics and Section III for Pure Mathematics candidates.

Use separate page for each question.

Section I

Q.1. Prove the Bernoulli's inequality
If $x > -1$, then $(1+x)^n \geq 1 + nx$, for all $n \in N$.

Q.2. Examine the convergence of the infinite series

$$1 + \frac{1}{(4)^{\frac{1}{3}}} + \frac{1}{(9)^{\frac{1}{3}}} + \frac{1}{(16)^{\frac{1}{3}}} + \dots$$

Q.3. Show that intersection of two normal subgroups is again a normal subgroup. Give an example to show that union of two subgroups need not to be a subgroup.

Q.4. (i) Prove that a metric space is a topological space.
(ii) Give an example of a topological space which is not a metric space.

Q.5. Find the general solution

$$y'' + e^{-x} y = 0.$$

Q.6. Find the dual basis of the following basis of $\mathbb{R}^3$: $\{(1,-2,3),(1,-1,1),(2,-4,7)\}$.

Q.7. Given that the velocity of a particle in rectilinear motion varies with the displacement x according to the equation $\dot{x} = bx^{-3}$ where b is a positive constant. Find the force acting on the particle as a function of x.

Q.8. Show that the given function $u(x,y) = \frac{x}{x^2+y^2}, z \neq 0$ is harmonic. Find the corresponding conjugate harmonic function $v(x,y)$ and construct the analytic function $f(z) = u + iv$.

Q.9. (a) What is the rank of the tensors representing:
(i) A straight line (ii) A scalar (iii) A metric on a sphere having unit radius.
(b) Find the contracted tensor components of F^a_b, where $F^{ac}_{db} = A^a_d B^c_b$ and

$$A^a_d = \begin{bmatrix} 0 & 2 \\ 1 & -5 \end{bmatrix}, \quad B^c_b = \begin{bmatrix} 0 & -1 & 0 \\ -3 & 0 & 7 \end{bmatrix}$$

Q.10. Apply the Simpson's formula to compute the integral of $\sin x$ between 0 and $\frac{\Pi}{2}$ from the values provided in the following table:

$\frac{h}{3}[f_0 + 4f_1 + \dots + f_n]$

x	0	$\frac{\Pi}{12}$	$\frac{\Pi}{6}$	$\frac{\Pi}{4}$	$\frac{\Pi}{3}$	$\frac{5\Pi}{12}$	$\frac{\Pi}{2}$
$sin x$	0.00000	0.25882	0.50000	0.70711	0.86603	0.96593	1.00000

Q.11. Consider the circular helix

$$\alpha(s) = (r\cos\omega s, r\sin\omega s, h\omega s)$$

where $\omega = \frac{1}{\sqrt{r^2+h^2}}$. Is α a unit speed curve? Compute the Frenet-Serret apparatus for the helix.

Q.12. Find the Laplace transform of $f(t) = \begin{cases} sin t & 0 < t < \Pi \\ 0 & t > \Pi \end{cases}$

Q.13. Show that a convergent sequence in a metric space is bounded.

Section II

Q.14. Use Simpson's $\frac{1}{3}$ rule to find the value of $\int_{0^\circ}^{360^\circ} \sin x\, dx$. Compare the result with exact solution.

Q.15. In a two dimensional incompressible flow the fluid velocity components are given by $u = x - 4y$, $v = -y - 4x$. Show that the flow satisfies the continuity equation and obtain the expression for the stream function. If the flow is of potential kind, obtain also the expression for the velocity potential.

Q.16. Solve the integral equation

$$u(x) = \sin x + 2\int_0^x \cos(x-t)u(t)dt.$$

Q.17. Prove that a displacement vector $\overline{u}$ can be written as

$$\overline{u} = \nabla\varphi + \nabla \times \overline{\Psi} \text{ where } \nabla \cdot \overline{\Psi} = 0.$$

Q.18. Write down the Maxwell's equations for homogenous isotropic and source-free medium. Also explain the terms their in. What are the constitutive relations.

Q.19. Using Dirac bra-ket algebra show that eigenstates corresponding to two distinct eigenvalues are orthogonal. Is the converse also true? Give an example.

Q.20. An initial frame F has coordinates (t, x, y, z) and frame F' has coordinates (t', x', y', z'), where F' is moving relative to F at speed $0.92c$ along the common $x - x'$ axis and the origins. Coinciding at $t = t' = 0$. If $x = 100m$, $y = 10m$, $z = 1m$ at $t = 5 \times 10^{-6}$secs, find the coordinates t', x', y', z' in F'. (Here c is the speed of light).

Section III

$|(x_m, x_n)| < K$

QUAID-I-AZAM UNIVERSITY

DEPARTMENT OF MATHEMATICS

M.Phil. Admission test Fall 2009

Time: 90 minutes

Dated: 24-08-2009

Note: Section I is compulsory, Section II is for Applied Mathematics and Section III for Pure Mathematics candidates.

Use separate page for each question.

Section I

Q.1. Prove that $\sqrt{2}$ is an irrational number.

Q.2. Define Riemann integral and show that every constant function on $[a,b]$ is in $R[a,b]$.

Q.3. Let G be a group of order 15. Show that every Sylow subgroup of G is normal in G.

Q.4. Solve $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \cos\omega t,\ 0 \le x \le \infty, 0 \le t \le \infty$

$u(0,t) = 0$, u is bounded as $x \to \infty$ $\quad \frac{\partial u}{\partial t}(x,0) = u(x,0) = 0.$

Q.5. Solve the following differential equation

$$(x+2)^2 y'' + (x+2)y' + y = 0$$

Q.6. Let $A = \left\{\begin{bmatrix} x & y \\ z & 0 \end{bmatrix} : x,y,z \in \mathbb{C}\right\}$, and $B = \left\{\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} : x,y \in \mathbb{C}\right\}$ be subspaces of $V = \left\{\begin{bmatrix} x & y \\ z & w \end{bmatrix} : x,y,z,w \in \mathbb{C}\right\}$. Show that $V = A+B$. What type of elements does $A \cap B$ contain.

Q.7. Find the residues at all the singular points of $f(z) = \frac{z^2}{(z^2+1)^2}$.

Q.8. Let $A_{\alpha\beta}$ be a symmetric tensor and $B_{\mu\nu}$ be a skew symmetric tensor. Work out the number of independent components of these tensors in 2 and 3 dimensions.

Q.9. Prove that a metric space is a topological space. Give an example of topological space which is not a metric space.

Q.10. Find an approximation to $\sqrt[3]{25}$ correct to three decimal positions using Regula falsi method. Perform only three iterations.

Q.11. Find the equation of tangent plane of the helix $x = (\cos t, \sin t, t)$ at $t = \frac{\pi}{2}$.

Q.12. Given that the velocity of a particle in rectilinear motion varies with the displacement x according to the equation $\dot{x} = \frac{b}{x^3}$ where b is a positive constant, find the force acting on the particle as a function of x.

Q.13. Let $(\mathbb{N}, \|.\|)$ be a normed space and (x_n) be a sequence in $\mathbb{N}$ such that $\| x_{n+1} - x_n \| < \frac{1}{2^n}$. Then (x_n) is a Cauchy sequence.

Q.21. If R is any commutative ring we define $GL(n, R)$ to the group of units of the matrix ring $M_n(R)$ in particular, $GL(n, \mathbb{Z})$ is the group of $n \times n$ matrices with integer coefficients whose inverses also have integer coefficients.

(i) Show that $GL(2, \mathbb{Z}) = \{A \in M_2(\mathbb{Z}) : \det(A) = \pm 1\}$.

(ii) Let A be an element of finite order in $GL(2, \mathbb{Z})$. Show that $ord(A)$ is $1, 2, 3, 4$ or 6.

Q.22. If the random variables X and Y have the joint probability density function given by $f_{X,Y}(x, y) = x + y \quad 0 < x < 1, \quad 0 < y < 1$ calculate the probability $P(X < Y)$.

Q.23. Let $T : H \to H$ and $W : H \to H$ be bounded linear operators on a complex Hilbert space H and $S = W^* T W$. Show that if T is self adjoint and positive, so is S.

Q.24. (a) Indicate true or false for a commutative ring R with identity.

i) Every ideal in R is a subring of R.

ii) Every maximal ideal is prime ideal in R.

iii) R/P is an integral domain $\Leftrightarrow$ P is prime ideal.

iv) $\mathbb{Z}$ is principal ideal domain.

v) $\mathbb{Z}_n$ is a finite field $\Leftrightarrow$ n is prime integer.

(b) Choose the correct one

	A	B	C
i) $1 + x + x^2$ has	roots in $\mathbb{Z}_2$	roots in $\mathbb{Z}_2$	neither
ii) $\mathbb{Z} \times \mathbb{Z}$ is	integral domain	field	neither
iii) $\mathbb{R} \times \mathbb{R}$ is	field	integral domain	neither
iv) $\mathbb{R}[X]$ is a	field	not a ring	neither
v) $X\mathbb{R}[X]$ is ...... ideal in $\mathbb{R}[X]$	Nil	trivial	neither.

Q.25. Find a relationship between $\ker \Pi$ and $\operatorname{Im} \phi$ for the sequences of homomorphisms of $\mathbb{Z}$-modules.

i) $0 \to 2\mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\Pi} \mathbb{Z}/2\mathbb{Z} \to 0$

ii) $0 \to \mathbb{Z}^k \xrightarrow{\phi} \mathbb{Z}^n \xrightarrow{\Pi} \mathbb{Z}^{n-k} \to 0 \quad n > k$

iii) $0 \to \mathbb{Z} \xrightarrow{\phi} \mathbb{Q} \xrightarrow{\Pi} \mathbb{Q} \to 0$

iv) $0 \to X\mathbb{R}[X] \xrightarrow{\phi} \mathbb{R}[X] \xrightarrow{\Pi} \mathbb{R} \to 0$

v) $0 \to \mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \oplus \mathbb{Q} \xrightarrow{\Pi} \frac{\mathbb{Z} \oplus \mathbb{Q}}{\mathbb{Z}} \to 0$.

Q.26. Show that an abelian group A is divisible if and only if A is injective as $\mathbb{Z}$-module.

Q.27. Show that union and intersection of two measurable sets is a measurable set.

$\boxed{\operatorname{Im} \phi \subset \ker \Pi}$

Section II

Q.14. Solve $\frac{d^2y}{dx^2} = y$ with $y(0) = 0$, $y(2) = 3.627$ using finite difference approximation and compare the result with exact solution.

Q.15. (a) Differentiate between body and surface forces.

(b) The velocity components in x and y directions are given as $u = \frac{2xy^3}{3} - x^2y$; $v = xy^2 - \frac{2yx^3}{3}$. Indicate whether the given velocity distribution is

(i) a possible field of flow.

(ii) not a possible field of flow.

Q.16. Solve the following integral equations

$$u(x) = \lambda \int_0^\pi (\cos^2 x \cos 2t + \cos 3x \cos^3 t) u(t) dt.$$

Q.17. Derive Hooke's law for homogeneous isotropic elastic bodies.

Q.18. For the case of electromagnetic phenomena in free space, what is the law that electric field $\vec{E}$ must satisfy.

Q.19. Give the time dependent Schrodinger equation. State it for a linear 1D harmonic oscillator to calculate its eigen energies and eigen states.

Q.20. A rod of rest length 10 meters move at half the speed of light parallel to its direction of motion. What will be its length according to an observer which is at rest relative to the rod.

Section III

Q.21. Let Q^x denote the group of rationals under multiplication, $K = \{1, -1\}$ and $H = < \frac{1}{2} >$. Then $HK = \{hk : h \in H \text{ and } k \in K\} = \{\pm h : h \in H\} = \{\pm(\frac{1}{2})^r : \text{for all } r \in Z\}$.

Show that $H/H \cap K$ is isomorphic to HK/K.

Q.22. Let R be a commutative ring with identity.

(i) Define a prime ideal and a primary ideal of R.

(ii) Prove that a prime ideal is also a primary ideal.

(iii) Give two examples of primary ideals in $\mathbb{Z}$ which are not prime ideals.

Q.23. Let A and B be closed subspaces of a Hilbert space H such that $A \perp B$. Then $A + B$ is a closed subspace of H.

Q.24. (i) Prove or disprove that quotient ring of an infinite ring is infinite.

(ii) Prove or disprove that quotient ring of an integral domain is an integral domain.

Q.25. Show that the p.d.f. of a normal random variable integrate to one.

Q.26. Show that a free module is projective.

Q.27. Show that the Lebesgue outer measure of an interval is its length.

(6)

QUAID-I-AZAM UNIVERSITY
Department of Mathematics
TEST FOR M.PHIL. ADMISSION/SPRING-2009

Time: 09:30 a.m. to 11.00 a.m. Dated: 29-01-2009

Note: Section I is compulsory, Section II-for Applied Mathematics and Section III is for Pure Mathematics candidates:

Section: I

Q.1 Find the local maximum, minimum or saddle point of the function

$$f(x,y) = (x^2+y)e^{y/2}$$

Q.2 (i). Let V be a vector space over the field F and W is its subspace. Prove that V/W is vector space over F.

(ii) $\mathbb{C}[X] = \{ f(x) = \sum_{i=0}^{n} z_i x^i \mid z_i \in \mathbb{C} \}$ is vector space over $\mathbb{R}$ and

$X\,\mathbb{C}[X] = \{ g(x) \in \mathbb{C}[X] \mid x \text{ divides } g(x) \}$ is subspace of $\mathbb{C}[X]$ over $\mathbb{R}$.

Q: Let $\beta(t)$ be a regular, non-unit speed curve in $\mathbb{R}^3$. Show that for torsion τ we have

$$\tau = [\dot{\beta}, \ddot{\beta}, \dddot{\beta}] / |\dot{\beta} \times \ddot{\beta}|^2$$

where the bracket denotes the scalar triple product.

Q: Let $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a metric on $\mathbb{R}$ defined by

$$d(x,y) = 0 \iff x = y$$

$$d(x,y) = |x| + |y| \iff x \neq y.$$

Find $S_2(2)$, $S_3(2)$ and $S_6(2)$. Prove or Disprove that $A = \{1, 2, 3, \cdots 20\}$ is an open set.

Q Show that with $W(x)=1$, the functions $\phi_i(x) = \cos(ix)$ $i = 0, 1, 2, \cdots$ are orthogonal over the interval $[-\pi, \pi]$. Construct the corresponding orthonormal set of functions.

Q. Show by an example that every bounded function need not be Riemann-integrable.

Q: Find the finite presentation of the symmetric group.

Q. Let $S_{\alpha\beta}$ be a symmetric tensor and $T^{\alpha\beta}$ be a skew symmetric tensor in an n-dimensional space.

(a) Work out the number of independent components of $S_{\alpha\beta}$.

(b) Work out the number of independent components of $T^{\alpha\beta}$.

(c) Show that $S_{\alpha\beta}T^{\alpha\beta} = 0$.

Q'. Let C_N denote the boundary of a square formed by the lines
$$x = \pm(N+\tfrac{1}{2})\pi, \qquad y = \pm(N+\tfrac{1}{2})\pi$$
where N is a positive integer, and let the orientation of C_N be counterclockwise.

Show that
$$\left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \leq \frac{16}{(2N+1)A}$$
where $|\sin z| \geq A$.

Q: A system consists of three particles, each of unit mass, with positions and velocities as follows

$\underline{r}_1 = \underline{i} + \underline{j}$ $\qquad$ $\underline{v}_1 = 2\underline{i}$

$\underline{r}_2 = \underline{j} + \underline{k}$ $\qquad$ $\underline{v}_2 = \underline{j}$

$\underline{r}_3 = \underline{k}$ $\qquad$ $\underline{v}_3 = \underline{i} + \underline{j} + \underline{k}$

(i) Find the position and velocity of the centre of mass.

(ii) Find the kinetic energy of the system.

Q: Find the inverse of $\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$ by LU decomposition method.

Q: Show That in an inner product space, $x \perp y$ if and only if $\|x + \alpha y\| \geq \|x\|$ for all scalars α.

Q: Solve the following "first order p.d.e"
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = (1+x)e^{y}$$

Q. (a) Compute the x-component of velocity u for one-dimensional steady compressible flow when $\rho = 1 + x^2$. Assume for $x = 0$; $u = 4$.

(b) Verify whether the following functions are valid potential functions:

(i) $\phi = A(x^2 - y^2)$ $\qquad$ (ii) $\phi = A\cos x$

(Spring 2008)

Q: Give the Klein - Gordon and the Dirac wave equations. Why the Klein - Gordon equation is not a valid wave equation, prove.

Q. A train of rest length 30 metres is moving with 60 percent of the speed of light. Calculate its length as it appears to an observer at rest on the platform.

Q: Write the Lagrangian for a simple harmonic oscillator. Using Lagrange's equations of motion deduce the equations of motion for the harmonic oscillator.

Q. Reduce the following integral equation to a differential equation

$$u(x) = \lambda \int_0^\infty e^{-|x-t|} u(t)\,dt$$

Q. Solve $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$ in the range $0 \le x \le 0.1$ Using modified Euler's method when $h = 0.1$

NA

Q One of the Maxwell's equations $\nabla \cdot \vec{D} = \rho$ can be derived from Gauss's law. Interpret this physically and derive Poisson's equation as well as Laplace's equation from it.

Q Let F be a field and $F^* = F \setminus \{0\}$. Show that $GL(3,F)/SL(3,F)$ is isomorphic to F^*.

Let R^2 be an inner product space and M be an independent subset of R^2. Find $M^\perp$.

Show that a commutative ring with identity is a field if and only if each ideal of it is prime.

Spring 2009

Q. Indicate true or false.

a) $\mathbb{Z}$ is not a module over $\mathbb{Q}$.

b) Every module is a vector space.

c) Every vector space is a Free module.

d) $\mathbb{Q}/\mathbb{Z}$ is module over $\mathbb{Z}$.

e) $\mathbb{Q}$ is module over $\mathbb{Z}$.

Show that a $\mathbb{Z}$-module is injective if and only if it is divisible.

(Spring 2007)

Show that a set of measure zero is measurable.

A number X is chosen at random from the series $4, 9, 14, 19, \ldots$ and another number Y is chosen from the series $1, 5, 9, 13, \ldots$. Each series has 100 terms. Find $P(X=Y)$.

1

QUAID – I – AZAMUNIVERSITY

DEPARTMENT OF MATHEMATICS

Test for M.Phil. admission fall 2008

Time: 9:30 a.m. to 11:00 a.m.

Dated: 19-08-2008

Note: Section I is compulsory, Section II is for Applied Mathematics candidates and Section III is for Pure Mathematics candidates.

Section I

Q.1 (a) If $0 < a < b$, determine $\lim_{n \to \infty} \left(\frac{a^{n+1} + b^{n+1}}{a^n + b^n} \right)$

(b) If $a > 0$, $b > 0$, show that

$$\lim_{n \to \infty} \left(\sqrt{(n+a)(n+b)} - n \right) = \frac{a+b}{2}$$

Q.2 State the Frenet-Serret apparatus for a unit speed curve. Let $\underline{\beta}(s)$ be a unit speed curve in $\mathbb{R}^3 \ni$ $K \neq 0$ and $\tau = 0$, then $\underline{\beta}(s)$ is part of a circle of radius $1/k$.

Q.3 Find moment of inertia of a uniform rod of mass 'm' and length 'a' about a line AB perpendicular to the rod and distant $\frac{a}{5}$ from one end of the rod. Then find the moment of inertia about a line perpendicular to the rod and distant $\frac{a}{3}$ from the line AB.

Q.4 Define the convergence of an infinite series and write a necessary condition for convergence of an infinite series. Further, examine the convergence of the series $1 + \frac{1}{4^{2/3}} + \frac{1}{9^{2/3}} + \frac{1}{16^{2/3}} + \cdots$

Q.5 A tensor T_{ab} in cartesian coordinates is given by

$$T_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Transform this tensor into polar coordinates (r, θ).

P.T.O.

Q.6 Using following tabular data approxima value of $f(2.5)$

x	-2	0	2	3
$f(x)$	0	2	10	29

Q.7 Give an example of two non-isomorphic binary operations on $\{1, 2, 3, 4\}$.

Q.8 (a) Find a linear mapping $f: R^2 \to R$ such that $\|f\| = 7$

(b) Show that $B(X, Y)$ bounded linear operators from a norm space X to a norm space Y cannot be finite

Q.9 Solve the following differential equation

$$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = (x-1)^2$$

Q.10 Solve $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$, $-\infty < x < \infty$, $t > 0$

$$u(x,0) = \begin{cases} 400, & -4 \le x \le 4 \\ 0, & 4 < |x| < +\infty \end{cases}$$

Q.11 (a) Show that $u(r,\theta) = \left(r + \frac{1}{r}\right)\cos\theta$ is harmonic function.

(b) Find the principal value of $\sqrt{1+C}$

Q.12 (a) Let X be a non-empty set and $x_0 \in X$; Let $\mathcal{T} = \{X\} \cup \{A \subseteq X : x_0 \notin A\}$. Show that $\mathcal{T}$ is a topology on X.

(b) Show that every neighbourhood in a co-finite topological space is an open set.

Q.13 (a) Let $W_1 = \{(a,b,0,0) : a, b \in R\}$, $W_2 = \{(0,0,c,0) : c \in R\}$, $W_3 = \{(0,0,0,d) : d \in R\}$. Prove that $R^4 = W_1 \oplus W_2 \oplus W_3$, where R^4 is a vector space over the field R.

(b) Let $T: R^2 \to R^2$ be defined by

$$T(a,b) = (a+2b, -2a+b) \text{ and } B = \{(1,0),(0,1)\}$$

Then illustrate Cayley-Hamilton Theorem.

Section II

14 Solve the following initial value problem

$$\frac{d^2u}{dx^2} = e^{2x} - \int_0^x e^{2(x-t)} \frac{du}{dt}\, dt$$

$u(0) = 0$, $u'(0) = 0.$

Q.15 The velocity components in a two-dimensional field are given by $V_r = \frac{\cos\theta}{r^2}$, $V_\theta = \frac{\sin\theta}{r^3}$ Find the equation of the streamline passing through the point $r = 2$, $\theta = \pi/2$

Q.16 Hookes law for homogeneous and isotropic elastic bodies is given by $\tau_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij}$ Derive this law ~~from~~ from generalized Hooke's law

$$\tau_{ij} = C_{ijkl}\, \epsilon_{kl}$$

Q.17 A rod of rest length 3 metres is moving with 90 percent of the speed of light (such that the length is in the direction of motion). Calculate the contraction in its length.

Q.18 Let the Hamiltonian operator for a system be defined as $\hat{H} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ 4 & 1 & 3 \end{bmatrix}$ write the operator as a combination of the outer product of the base states $\{ \langle u_i |, i = 1,2,3 \}$. What are the allowed energy eigenvalues for this system.

Q.19 Find Hamilton's Equations of a particle which executes simple Harmonic motion

Q.20 Solve $\frac{dy}{dx} = yz + x$, $\frac{dz}{dx} = xz + y$ given that $y(0) = 1$, $z(0) = -1$. By Runge - Kutta Method of fourth order.

Section III

Q.21 Let $T: X \to X$ be a bounded linear operator on a complex inner product space X. Show that if $\langle Tx, x \rangle = 0$ for all $x \in X$ then $T = 0$.

Q.22 Show that the mean of the cauchy distribution does not exist.

Q.23 Let R be a commutative ring with identity and P an ideal in R. Prove that R/P is an integral domain if and only if P is prime ideal in R. Why a maximal ideal in R is a prime ideal?

Q.24 Indicate True or false

✓(i) Every additive abelian group is module over $\mathbb{Z}$.

×(ii) $\mathbb{Z}$ is not a module over $\mathbb{Q}$.

✓(iii) $\mathbb{Q}/\mathbb{Z}$ is module over $\mathbb{Z}$.

(iv) $\mathbb{Q}$ is not a module over $\mathbb{Z}$.

(v) $R[x]$ is finitely generated module over R.

Q.25 Show that, in their natural presentations on the appropriate projective lines, $PSL(2,5) = A_5$.

Q.26 Show that an abelian group under addition is divisible iff it is injective as $\mathbb{Z}$-module.

Q.27 Show that every set of measure zero is measurable.

$\mathbb{Q} \times \mathbb{Z} \not\to \mathbb{Z}$

Good Luck

QUAID-I-AZAM UNIVERSITY

DEPARTMENT OF MATHEMATICS

Test for M. Phil. admission spring 2008

Time: 9:30 a.m. to 11:00 a.m.
Dated: 4-02-2008.
Note: Section I is Compulsory, Section II is for Applied Mathematics Candidates and Section III is for Pure Mathematics Candidates.

Section I

Q.1. Evaluate the following integral

$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)dy\, dx.$$

Q.2. Let $\beta(s)$ be a unit speed curve in $\mathbb{R}^3$ such that $\kappa \neq 0$ is a constant and $\tau = 0$. Then Prove that $\beta(s)$ is a circle of radius $\frac{1}{\kappa}$.

Q.3. Define moment of a force. Find at any time t the moment of the force $\underline{F} = t\underline{i} + t^2\underline{j} + (2t^3 - 1)\underline{k}$ acting on a particle of mass m whose position vector relative to an origin O is given by $\underline{r} = 2\underline{i} + 3\underline{j} - 5\underline{k}$.

Q.4. Find the contracted tensor components of G^a_b, where $G^{ac}_{db} = A^a_d B^c_b$ and $A^a_d = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$, $B^c_b = \begin{bmatrix} -2 & -1 \\ 4 & 3 \end{bmatrix}$.

Q.5. a) Determine residue of $f(z) = \frac{1}{z(e^z-1)}$.

b) Determine the number of zeros, counting multiplicities of the polynomial $2z^5 - 6z^2 + z + 1$ in the annulus $1 \leq |z| \leq 2$.

Q.6. Let G be a group and $N \leq G$, $K \leq G$ such that $N \vartriangle K$. Then Prove that $K / N \vartriangle G / N$.

Q.7. Show that in a normed space every convergent sequence is bounded. Give an example to show that converse is not always true.

Q.8. a) Verify that $\mathbb{Q} \oplus \mathbb{R}$ is a vector space over the field $\mathbb{Q}$ but not a vector space over $\mathbb{R}$.

b) What are the dimensions of quotient spaces $\frac{\mathbb{R}\oplus\mathbb{C}}{\mathbb{R}}$ and $\frac{\mathbb{C}^n}{\mathbb{R}}$.

c) Define algebra and division algebra.

Q.9. Let X be a non empty set and $\Im$ consists of empty set and all those subsets of X whose complement is finite. Show that $\Im$ is a topology on X.

Q.10. Evaluate $\int_0^{90^\circ} \cos x\, dx$ by using Simpson's $\frac{1}{3}$ rule for $n = 6$.

Q.11. Solve $y'' - 2y' + y = 10e^{-2x}\cos x$ by the method of undetermined coefficients — Annihilator approach.

Q.12. Show that the function $f(x)$ defined as follows

(9)

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

is nowhere differentiable. What can you say about the integrability of this function.

Q.13. Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0$$
$$u(0,t) = 0, \quad t > 0,$$
$$u(x,0) = f(x), \quad 0 < x < \infty$$

and u, $\frac{\partial u}{\partial x}$, both tend to zero as $x \to \infty$.

Section II

Q.14. A velocity field is given by $u = 3x^2$, $v = 2x$, $w = 0$ in arbitrary units. Is the flow is steady or unsteady? Is it One, two or three dimensional? At $(x, y, 0) = (2, 1, 0)$ compute the total acceleration.

Q.15. Calculate the energy eigenvalues and eigenstates for a one dimensional harmonic oscillator.

Q.16. State the two basic postulates of the special theory of relativity. Use them to deduce the Lorentz transformation between coordinates (x, y, z, t) and (x', y', z', t') for uniform relative velocity v along the x-direction.

Q.17. The Lagrangian function describing the dynamics of a particle is given by $L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}(q_1^2 + q_2^2)$. Find the Hamilton equations of motion for this system.

Q.18. Find the resolvent kernel to solve the following equation

$$u(x) = f(x) + \lambda \int_0^x e^{x-t} u(t)dt.$$

Q.19. U Runge-Kutta method of fourth order to solve
$y' = xy + y^2$, $y(0) = 1$ for $y(0.1)$.

Q.20. Write the Maxwell's equations for electromagnetic medium and explain all relevant terms.

Section III

Q.21. Prove that a group of order 15 is not simple.

Q.22. Consider $X = \mathbb{R}^2$ be an inner product space and let $M = \{(1, 2)\}$. Find $M^{\perp}$

Q.23. Let R be a commutative ring with identity and I be an ideal in R. Prove that R/I is an integral domain iff I is prime ideal.

Q.24. Indicate true or false.

a) Every module is a vector space.

b) Every vector space is a module.

c) An additive abelian group is a module over $\mathbb{Z}$.

d) An additive abelian group is a module over $\mathbb{Q}$.

e) Every primary ideal of a commutative ring R with identity is prime ideal.

Q.25. Show that every free module is projective.

Q.26. Show that outer measure of an interval is its length.

Q.27. Show that if $P[a \leq x \leq b] = 1$ then $a \leq E[x] \leq b$.

GOOD LUCK